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A remark on representation of dependent random
variables as functions of independent ones

by

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This note contains a remark which, although an only slightly less general remark has been made by P. Lévy ¹⁾, seems not to be generally known. A few simple consequences concerning stochastic processes are also mentioned.

§ 1

As usual two random variables $\underline{x}$ and $\underline{y}$ ²⁾ are said to be stochastically dependent if they have a common distribution. A particular case is the one where they are functionally dependent, e.g. where $\underline{y}$ is a function of $\underline{x}$: $\underline{y} = \varphi(\underline{x})$, $\varphi(\underline{x})$ being a measurable function. A more general case is the one where the ordinary function $\varphi(\underline{x})$ of one variable is replaced by a random function, e.g. a function $\varphi(\underline{x}, \theta, \theta_1, \dots)$ depending on one or more or even an infinity of parameters for which random variables $\underline{a}_1, \underline{a}_2, \dots$ which are stochastically independent of $\underline{x}$ are substituted :

$$\underline{y} = \varphi(\underline{x}, \underline{a}_1, \underline{a}_2, \dots)$$

At first sight one might assume this still to be a special case. This, however, is not so, and, moreover, one parameter is sufficient.

In fact, let $\underline{x}$ and $\underline{y}$ be stochastically dependent and let us assume that the conditional distribution function $F(\underline{y}|\underline{x})$ of $\underline{y}$, given $\underline{x}$, exists spr ³⁾ 0, and is measurable with respect to $\underline{x}$ and $\underline{y}$ separately as well as simultaneously. We take here the distribution function continuous from the left:

$$(1) \quad F(\underline{y}|\underline{x}) \stackrel{\text{def}}{=} P\{\underline{y} < \underline{y}|\underline{x}\} \quad 4)$$

We introduce the discontinuous function

$$(2) \quad p(\underline{y}|\underline{x}) \stackrel{\text{def}}{=} P\{\underline{y} = \underline{y}|\underline{x}\}$$

which vanishes everywhere, except on a denumerable set. Moreover, we put for $0 \leq a \leq 1$

$$(3) \quad \varphi(\underline{x}, a) \stackrel{\text{def}}{=} \sup\{\underline{y} | F(\underline{y}|\underline{x}) < a\}$$

1) P. Lévy, Théorie de l'addition des variables aléatoires, 1e ed., Paris, 1937, p. 71-73, 122-123.

2) Random variables are denoted by underlined symbols.

3) spr α (salva probabilitate α) means: except for a probability α . Hence spr 0 is equivalent with: almost surely.

4) $\stackrel{\text{def}}{=}$ denotes an equality defining the left hand member.

Then, with $\underline{a}$ stochastically independent of $\underline{x}$ and $\underline{y}$ and $H(0,1)$ (i.e. homogeneously distributed on $(0,1)$); putting

$$(4) \quad \underline{a} \stackrel{\text{def}}{=} F(\underline{y}|\underline{x}) + \underline{a} P(\underline{y}|\underline{x})$$

we have

$$P\{\underline{a} \leq c | \underline{x}\} = c$$

for every c with $0 \leq c \leq 1$ and almost every $\underline{x}$, i.e. $\underline{a}$ is stochastically independent of $\underline{x}$ and

$$(5) \quad \underline{y} = \varphi(\underline{x}, \underline{a}) \quad \text{spr } 0.$$

Hence we have:

I For any pair of stochastically dependent random variables $\underline{x}, \underline{y}$ a measurable function $\varphi(\underline{x}, \underline{a})$ of two variables can be found such that (5) holds with $\underline{a} \sim H(0,1)$ and stochastically independent of $\underline{x}$.

We notice that

- a) the assumption that $\underline{x}$ is one-dimensional enters nowhere.
- b) the case of stochastic independence of $\underline{x}$ and $\underline{y}$ enters as the special case where φ does not depend on its first argument.
- c) the case of functional dependence of $\underline{y}$ on $\underline{x}$ enters as the special case, where φ does not depend on its second argument.
- d) the above proof is identical with the one (obtained by putting $\underline{x} = 0, \text{spr } 0$) for the known fact that any one-dimensional random variable $\underline{y}$ is a function of a $H(0,1)$ distributed random variable.

Using these facts we obtain:

II If $\underline{x}_1, \dots, \underline{x}_n$ have a common distribution function and are one-dimensional, then for every integer k ($1 \leq k \leq n$) a measurable function φ_k of k variables exists, such that for all $k \in \{1, \dots, n\}$

$$(6) \quad \underline{x}_k = \varphi_k(\underline{a}_1, \dots, \underline{a}_k) \quad \text{spr } 0$$

where $\underline{a}_1, \dots, \underline{a}_n$ are $H(0,1)$ and (completely) stochastically independent.

§ 2

Let $\underline{x}(t)$ be a stochastic process on a set T of elements t . If T is finite or enumerable the result II may be applied. In the latter case, taking for T the set of positive integers, we get for each $t \in T$ $\underline{x}(t) = \varphi(t, \underline{a}_1, \dots, \underline{a}_t)$, the $\underline{a}_t$ being all $H(0,1)$ and stochastically independent.

Assume now, for an arbitrary T , that a topology is defined in T , that $T_0 \stackrel{\text{def}}{=} \{t_1, t_2, \dots\}$ is everywhere dense in T , and that

the process is everywhere continuous $\text{spr } 0$. We again have for every $t_k \in T_0$ a representation of the form (6) for $\underline{x}(t_k)$. For any $t \in T$ there is a subsequence $t_{\nu_1}, t_{\nu_2}, \dots$ converging towards t , and

$$(7) \quad \underline{x}(t) = \underline{x}(t_{\nu_1}) + \sum_{n=1}^{\infty} (\underline{x}(t_{\nu_{n+1}}) - \underline{x}(t_{\nu_n})) \text{ spr } 0, \text{ so that for every } t \in T \text{ a representation of the form}$$

$$(8) \quad \underline{x}(t) = \varphi(t, \underline{u}_1, \underline{u}_2, \dots)$$

exists, the function φ being measurable with respect to the $\underline{u}_n$, but not necessarily so with respect to t . A similar result holds if the condition of absolute continuity is replaced by an other one, admitting a representation $\text{spr } 0$ of $\underline{x}(t)$ as a function of the $\underline{x}(t_{\nu})$.

In the most general case measure theoretic difficulties of the customary type arise. If, however, $\underline{x}(t)$ for every $t \in T$ has a conditional probability distribution, given $\underline{x}(t_1), \underline{x}(t_2), \dots$, I again can be applied, so that $\underline{x}(t)$ for $t \in T - T_0$ can be expressed by means of the $\underline{u}(t_n) (= \underline{u}_n)$ and new independent $H(0,1)$ variables $\underline{u}(t)$. It seems therefore worthwhile to try to find out, under exactly which conditions a representation of the form

$$(9) \quad \underline{x}(t) = \varphi(t, \underline{u}(s))$$

exists, where s runs through a set $S \subset T$ and $\underline{u}(s)$ are $H(0,1)$ for all $s \in S$, and independent in the sense that for any function $f(s)$ on S such that $0 \leq f(s) \leq 1$ and $f(s) = 1$ except for at most at a countable number of elements $s_1, s_2, \dots \in S$,

$$P \{ 0 \leq \underline{u}(s) \leq f(s) \} = \prod_{i=1}^{\infty} f(s_i).$$